Optimal Approximation of Dynamical Systems with Rational Krylov Methods

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Outline

1. Introduction and Problem Statement
2. Motivating Examples
3. Rational Krylov-Interpolation Framework
4. An Iterative Rational Krylov Algorithm
5. Inexact Solves in Krylov-based Model Reduction

Introduction

- Consider an n^{th} order single-input/single-output system $\mathbf{G}(s)$:

$$\mathbf{G}(s) : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{c}\,\mathbf{x}(t) \end{cases} \quad \Leftrightarrow \quad \begin{aligned} \mathbf{G}(s) &= \mathbf{c}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{b} \\ &= \frac{\mathbf{n}(s)}{\mathbf{d}(s)} \end{aligned}$$

- $\mathbf{u}(t) \in \mathbb{R}$: input, $\mathbf{x}(t) \in \mathbb{R}^n$: state, $\mathbf{y}(t) \in \mathbb{R}$: output

- $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$. Will assume $\Re(\lambda_i(\mathbf{A})) < 0$

- Need for improved accuracy $\Longrightarrow$ Include more details in the modeling stage

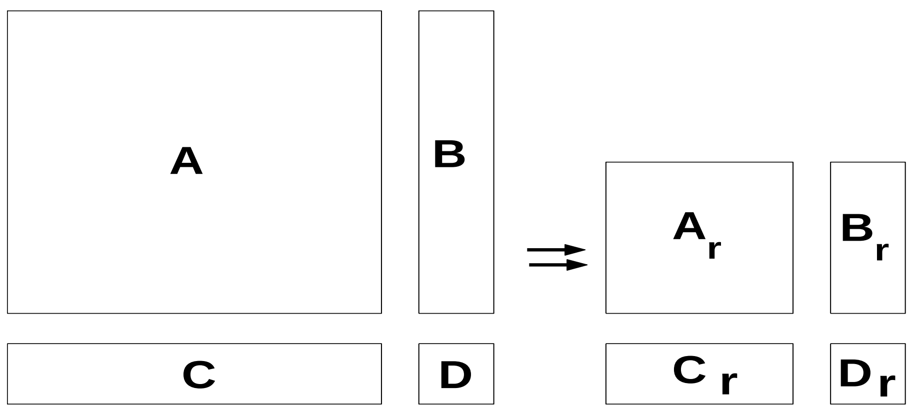
- In many applications, n is quite large, $n \approx \mathcal{O}(10^6, 10^7)$,

- Untenable demands on computational resources $\Longrightarrow$

Model Reduction Problem: Find

$$\begin{aligned}\dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\,\mathbf{x}_r(t) + \mathbf{b}_r\,\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{c}_r\,\mathbf{x}_r(t)\end{aligned} \quad \Leftrightarrow \quad \mathbf{G}_r(s) = \mathbf{c}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$$

- where $\mathbf{A}_r \in \mathbb{R}^{r\times r}$, $\mathbf{b}_r, \mathbf{c}_r^T \in \mathbb{R}^r$, with $r \ll n$ such that
    1. $\|\mathbf{y} - \mathbf{y}_r\|$ is *small.*
    2. The procedure is *computationally efficient.*



- $\mathbf{G}_r(s)$: used for simulation or designing a reduced-order controller

Model reduction of Serkan from n=3 down to r=2

Cascades, Blacksburg, VA

- Model reduction through projection: a unifying framework.

- Construct $\mathbf{\Pi} = \mathbf{V}\mathbf{Z}^T$, where $\mathbf{V}, \mathbf{Z} \in \mathbb{R}^{n\times r}$ with $\mathbf{Z}^T\mathbf{V} = \mathbf{I}_r$:

$$\dot{\mathbf{x}}_r = \underbrace{\mathbf{Z}^T\mathbf{A}\mathbf{V}}_{:=\mathbf{A}_r}\mathbf{x}_r(t) + \underbrace{\mathbf{Z}^T\mathbf{b}}_{:=\mathbf{b}_r}\mathbf{u}(t), \quad \mathbf{y}_r(t) = \underbrace{\mathbf{c}\mathbf{V}}_{:=\mathbf{c}_r}\mathbf{x}_r(t)$$

$$\boxed{\text{What is the approximation error } \mathbf{e}(t) := \mathbf{y}(t) - \mathbf{y}_r(t)?}$$

- $\mathbf{G}(s)$: Associate a convolution operator $\mathcal{S}$:

$$\mathcal{S} : \mathbf{u}(t) \mapsto \mathbf{y}(t) = (\mathcal{S}\mathbf{u})(t) = (\mathbf{g} \star \mathbf{u})(t) = \int_{-\infty}^{t} \mathbf{g}(t-\tau)\mathbf{u}(\tau)d\tau.$$

- $\mathbf{g}(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b}$ for $t \geq 0$: *Impulse response*.

- Transfer function: $\mathbf{G}(s) = (\mathcal{L}\mathbf{g})(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$.

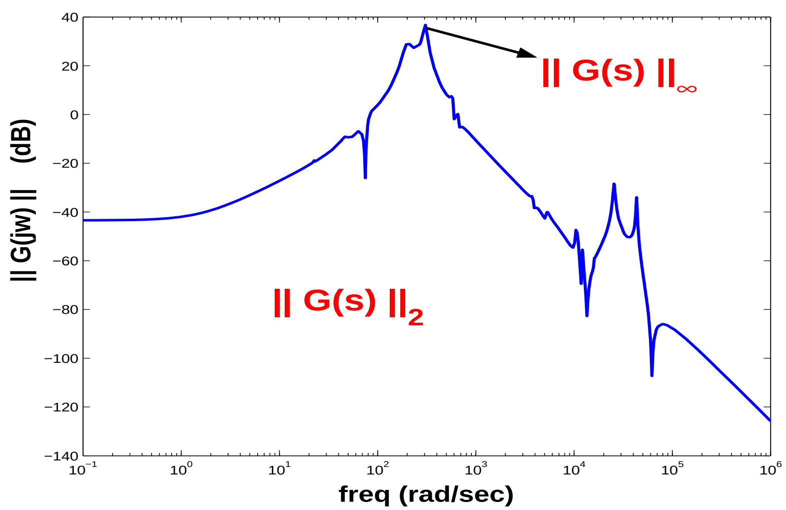
The $\mathcal{H}_\infty$ Norm : 2-2 induced norm of $\mathcal{S}$:

$$\|\mathbf{G}(s)\|_{\mathcal{H}_\infty} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathcal{S}u\|_2}{\|\mathbf{u}\|_2} = \sup_{w\in\mathbb{R}} \|\mathbf{G}(\jmath w)\|_2$$

$\|\mathbf{G}-\mathbf{G}_r\|_\infty =$ Worst output error $\|\mathbf{y}(t)-\mathbf{y}_r(t)\|_2 \quad \forall \quad \|\mathbf{u}(t)\|_2 = 1.$

$\boxed{\textbf{The } \mathcal{H}_2 \textbf{ Norm :}}$ $\mathcal{L}_2$ norm of $\mathbf{g}(t)$ in time domain:

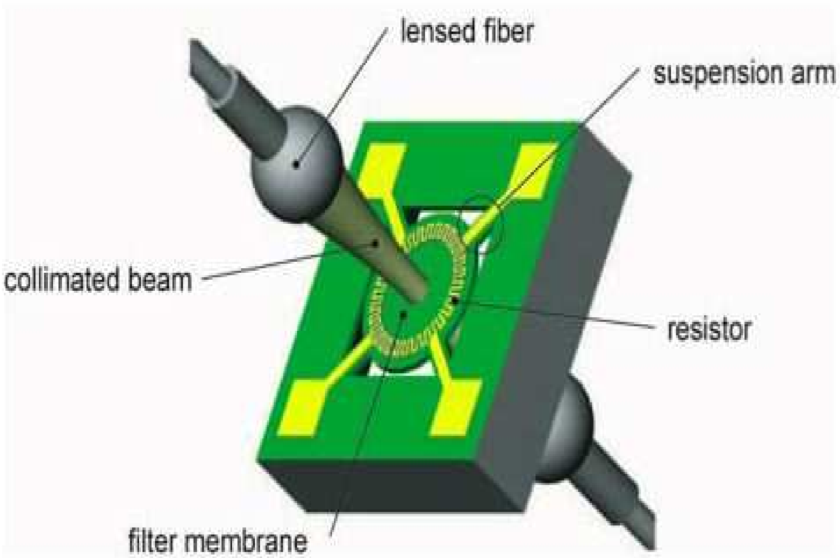
$$\|\mathbf{G}(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \mathrm{trace}[\mathbf{g}^T(t)\mathbf{g}(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{trace}[\mathbf{G}^*(\jmath w)\mathbf{G}(\jmath w)]dw$$



Motivating Example: Simulation

A Tunable Optical Filter: (Data: D. Hohlfeld, T. Bechtold, and H. Zappe)

- An optical filter, tunable by thermal means.



- Silicon-based fabrication.
- The thin-film filter: membrane to improve thermal isolation
- Wavelength tuning by thermal modulation of resonator optical thickness

- The device features low power consumption, high tuning speed and excellent optical performance.

Modeling:

- A simplified thermal model to analyze/simulate important thermal issues: 2D model and 3D model
- Meshed and discretized in ANSYS 6.1 by the finite element methods
- The Dirichlet boundary conditions at the bottom of the chip.
- A constant load vector corresponding to the constant input power of of 1 mW for 2D model and 10 mW for 3D model
- The output nodes located in the membrane

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{c}\,\mathbf{x}(t)$$

- 2D: $n = 1668$, $nnz(\mathbf{A}) = 6209$, $nnz(\mathbf{E}) = 1668$
- 3D: $n = 108373$, $nnz(\mathbf{A}) = 1406808$, $nnz(\mathbf{E}) = 1406791$

Motivating Example: Control

Optimal Cooling of Steel Profiles in a Rolling Mill :

Data: Peter Benner

- Different steps in the production process require different temperatures of the raw material.
- To achieve high throughput, reduce the temperature as fast as possible to the required level before entering the next production phase.
- Cooling process by spraying cooling fluids on the surface
- Must be controlled so that material properties, such as durability or porosity, stay within given quality standards
- Modeled as boundary control of a two dimensional heat equation.
- A finite element discretization results in

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathbf{c}\,\mathbf{x}(t).$$

- $n = 79,841$: $nnz(\mathbf{A}) : 553921$, $nnz(\mathbf{E}) : 554913$

Model Reduction via Interpolation

Rational Interpolation: Given $\mathbf{G}(s)$, find $\mathbf{G}_r(s)$ so that

$\mathbf{G}_r(s)$ *interpolates* $\mathbf{G}(s)$ and certain number of its derivatives at selected frequencies σ_k in the complex plane

$$\left.\frac{(-1)^j}{j!}\frac{d^j\mathbf{G}(s)}{ds^j}\right|_{s=\sigma_k} = \left.\frac{(-1)^j}{j!}\frac{d^j\mathbf{G}_r(s)}{ds^j}\right|_{s=\sigma_k}, \quad \begin{array}{l}\text{for } k=1,\ldots,K, \\ \text{and } j=1,\ldots,J\end{array}$$

- $$\left.\frac{(-1)^j}{j!}\frac{d^j\mathbf{G}(s)}{ds^j}\right|_{s=\sigma_k} = \mathbf{c}(\sigma_k\mathbf{I}-\mathbf{A})^{-(j+1)}\mathbf{b}:$$
  $$= j^{\text{th}} \text{ **moment** of } \mathbf{G}(s) \text{ at } \sigma_k.$$

Why to choose model reduction via rational interpolation?

- Generically, any reduced model $\mathbf{G}_r(s)$ can be obtained via interpolation.
- Interpolation points = Zeroes of $\mathbf{G}(s) - \mathbf{G}_r(s)$.
- BUT:

Prob-1: What is a good selection of interpolation points?

- Similar to polynomial approximation of complex functions.
- Recall: Trying to match the moments: $\mathbf{c}(\sigma_k \mathbf{I} - \mathbf{A})^{-(j+1)}\mathbf{b}$
- Moments are extremely ill-conditioned

Prob-2: Construct $\mathbf{G}_r(s)$ without explicit moment computation

- Prob-2 easier to tackle using rational Krylov framework (Skelton *et al.* [1987], Grimme [1997]):

- Given r interpolation points: $\{\sigma_i\}_{i=1}^{r}$
- Set $\mathbf{V} = \text{Span}\left[(\sigma_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}, \cdots, (\sigma_r\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}\right]$, and
- $\mathbf{Z} = \text{Span}\left[(\overline{\sigma_1}\,\mathbf{I}-\mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I}-\mathbf{A}^T)^{-1}\mathbf{c}^T\right]$, $\mathbf{Z}^T\mathbf{V} = \mathbf{I}_r$.
- $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V}$, $\mathbf{b}_r = \mathbf{Z}^T\mathbf{b}$, $\mathbf{c}_r = \mathbf{c}\mathbf{V}$

$$\Longrightarrow \boxed{\mathbf{G}(\sigma_i) = \mathbf{G}_r(\sigma_i), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}(s)\Big|_{s=\sigma_i} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}_r(s)\Big|_{s=\sigma_i}}$$

- Moment matching without explicit moment computation
- Still to answer: How to choose σ_i?
- $\sigma_i = -\lambda_i(\mathbf{A})$ (Antoulas/G [2003]). Effective but not optimal.
- Does there exist an optimal selection?

Optimal $\mathcal{H}_2$ approximation

Problem: Given a stable dynamical system $\mathbf{G}(s)$, find a reduced model $\mathbf{G}_r(s)$ that satisfies

$$\mathbf{G}_r(s) = \arg \min_{\substack{\deg(\hat{\mathbf{G}}) = r \\ \hat{\mathbf{G}} \,:\, \text{stable}}} \left\| \mathbf{G}(s) - \hat{\mathbf{G}}(s) \right\|_{\mathcal{H}_2}.$$

- Existence of a global minimal:
  - Exists in the SISO case
  - Not known for the MIMO case
- General approach: Find $\mathbf{G}_r(s)$ that satisfies first-order necessary conditions: Wilson [1970], Meier and Luenburger [1967], Hyland and Bernstein [1985], Yan and Lam [1999], ...

Framework of Wilson [1970]

- Given $\mathbf{G}_r(s) = \mathbf{c}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$, define the error system

$$\mathbf{G}_e(s) := \mathbf{G}(s) - \mathbf{G}_r(s) = \mathbf{c}_e(s\mathbf{I} - \mathbf{A}_e)^{-1}\mathbf{b}_e$$

- Let $\mathbf{P}_e$ and $\mathbf{Q}_e$ be the error gramians:

$$\mathbf{A}_e\mathbf{P}_e + \mathbf{P}_e\mathbf{A}_e^T + \mathbf{b}_e\mathbf{b}_e^T = 0, \quad \mathbf{Q}_e\mathbf{A}_e + \mathbf{A}_e^T\mathbf{Q}_e + \mathbf{c}_e^T\mathbf{c}_e = 0$$

- $\mathbf{P}_e = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} \end{bmatrix}, \quad \mathbf{Q}_e = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{12}^T & \mathbf{Q}_{22} \end{bmatrix}$

- $\|\mathbf{G}_e(s)\|_{\mathcal{H}_2}^2 = \mathbf{c}_e\mathbf{P}_e\mathbf{c}_e^T$: $\implies$ First-order necessary conditions:

$$\left.\begin{aligned} \mathbf{P}_{12}^T\mathbf{Q}_{12} + \mathbf{P}_{22}\mathbf{Q}_{22} &= 0 \\ \mathbf{Q}_{12}^T\mathbf{b} + \mathbf{Q}_{22}\mathbf{b}_r &= 0 \\ \mathbf{c}_r\mathbf{P}_{22} - \mathbf{c}\mathbf{P}_{12} &= 0. \end{aligned}\right\}$$

- Equivalently, $\mathbf{V} = \mathbf{P}_{12}\mathbf{P}_{22}^{-1}$, $\mathbf{Z} = -\mathbf{Q}_{12}\mathbf{Q}_{22}^{-1}$ and

$$\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V},\ \mathbf{b}_r = \mathbf{Z}^T\mathbf{b},\quad \mathbf{c}_r = \mathbf{c}\mathbf{V}.$$

- $\mathcal{H}_2$ Iteration:

1. Choose an initial $\mathbf{G}_r(s) = \mathbf{c}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$.
2. Compute $\mathbf{P}_e$ and $\mathbf{Q}_e$
3. Define $\mathbf{V} = \mathbf{P}_{12}\mathbf{P}_{22}^{-1}$, $\mathbf{Z} = -\mathbf{Q}_{12}\mathbf{Q}_{22}^{-1}$
4. Let $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V},\ \mathbf{b}_r = \mathbf{Z}^T\mathbf{b},\quad \mathbf{c}_r = \mathbf{c}\mathbf{V}$.
5. Return to Step 1.

- Two Lyapunov equations at each step.
- Similar framework by Hyland and Bernstein [1985]

Framework of Meier and Luenberger [1967]

- Let $\mathbf{G}_r(s) = \mathbf{c}_r(s\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$ solves the optimal $\mathcal{H}_2$ problem
- Let $\hat{\lambda}_i = \lambda_i(\mathbf{A}_r)$, i.e. the Ritz values.
- First-order conditions:

$$\boxed{\mathbf{G}(-\hat{\lambda}_i) = \mathbf{G}_r(-\hat{\lambda}_i), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}(s)\Big|_{s=-\hat{\lambda}_i} = \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}_r(s)\Big|_{s=-\hat{\lambda}_i}}$$

- Match the first two moments at the mirror images of the Ritz values.
- First-order conditions as interpolation.

$$\Downarrow$$

- Rational Krylov Framework

Theorem: The two frameworks are equivalent.

Proof: Starting point for Lyapunov $\rightarrow$ Interpolation Framework:

Lemma: (Gallivan *et al.* [2004], Antoulas/Sorensen [2002])
Let $\mathbf{V}$ solves $\mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}_r^T + \mathbf{b}\mathbf{b}_r^T = 0$. Then,

$$\mathrm{Ran}(\mathbf{V}) = \mathrm{Span}\left[(-\hat{\lambda}_1\,\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},\ \cdots\ ,(-\hat{\lambda}_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right].$$

Starting point for Interpolation $\rightarrow$ Lyapunov Framework: Model reduction via rational Krylov projection.

- For the $\mathcal{H}_2$ problem, simply set $\boxed{\sigma_i = -\hat{\lambda}_i}$
- $\hat{\lambda}_i$ NOT known a priori $\Longrightarrow$ Needs iterative rational steps

An Iterative Rational Krylov Algorithm (IRKA):
(G, Beattie, Antoulas [2004])

1. Choose σ_i for $i = 1, \ldots, r$.
2. $\mathbf{V} = \text{Span}\left[(\sigma_1 \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \cdots, (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right]$,
3. $\mathbf{Z} = \text{Span}\left[(\overline{\sigma_1}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T\right]$, $\mathbf{Z}^T\mathbf{V} = \mathbf{I}_r$.
4. while [relative change in σ_j] $> \epsilon$
   (a) $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V}$,
   (b) $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r)$ for $i = 1, \ldots, r$
   (c) $\mathbf{V} = \text{Span}\left[(\sigma_1\,\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \ \cdots \ , (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\right]$.
   (d) $\mathbf{Z} = \text{Span}\left[(\overline{\sigma_1}\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T, \cdots, (\overline{\sigma_r}\,\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{c}^T\right]$, $\mathbf{Z}^T\mathbf{V} = \mathbf{I}_r$.
5. $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V}, \quad \mathbf{b}_r = \mathbf{Z}^T\mathbf{b}, \quad \mathbf{c}_r = \mathbf{c}\mathbf{V}$

- Upon convergence, first-order conditions satisfied via Krylov projection framework, no Lyapunov solvers

- No methods guarantee convergence to global minimum.
- **Question:** Global minimum of a *restricted* $\mathcal{H}_2$ minimization problem?

Corollary: (Gaier 1980)

Given stable $\mathbf{G}(s)$, and the stable reduced poles $\alpha_1, \ldots, \alpha_r$, define

$$\widehat{\mathbf{G}}(s) := \frac{\beta_0 + \beta_1 s + \cdots + \beta_r s^r}{(s - \alpha_1) \ldots (s - \alpha_r)}.$$

Then $\|\mathbf{G}(s) - \widehat{\mathbf{G}}(s)\|_{\mathcal{H}_2}$ is minimized if and only if

$$\mathbf{G}(s) = \hat{\mathbf{G}}(s) \quad \text{for} \quad s = -\overline{\alpha}_1, -\overline{\alpha}_2, \ldots, -\overline{\alpha}_r.$$

- Upon convergence, **IRKA** *minimizes* the $\mathcal{H}_2$ norm of the error system among all possible reduced models having the same reduced poles $\widehat{\lambda}_j$.

Convergence ?

- Understood better and better every day !!!
- A fixed point iteration:

$$\left\{{\sigma_i}^{(k+1)}\right\} = \mathbf{f}\left(\left\{{\sigma_i}^{(k)}\right\}\right) \;\Rightarrow\; \mathbf{\Pi}^{(k+1)} = \mathbf{h}\left(\mathbf{\Pi}^{(\mathbf{k})}\right)$$

- Usual outcome is (numerical) convergence in $4 - 5$ steps
- Convergence failure in rare circumstances.
- Newton Iteration Framework:
  - Jacobian $\mathbf{J}$: Sensitivity of $\lambda_i(\mathbf{A}_r)$ wrt $\{\sigma_i\}$
  - Requires solving an $r \times r$ generalized eigenvalue problem

$$\{\sigma_i\}^{(k+1)} = \{\sigma_i\}^{(k)} - (\mathbf{I} + \mathbf{J})^{-1}\left(\{\sigma_i\}^{(k)} + \{\lambda_i(\mathbf{A}_r)\}^{(k)}\right).$$

Stability ?

- $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}\mathbf{V}$ nonnormal reduced order model
  $\longrightarrow$ Reduced order stability not guaranteed in general.
- **But**, very hard to force convergence to unstable model
  (occasional unstable models can occur at intermediate stages)
- Fairly robust with respect to initial shift selection.
- Gugercin [CDC-2005]: Replace $\mathbf{Z}$ by $\mathbf{Q}\mathbf{V}(\mathbf{V}^T\mathbf{Q}\mathbf{V})^{-1}$ where

$$\mathbf{A}^T\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{c}^T\mathbf{c} = 0.$$

$\longrightarrow$ implies stability.

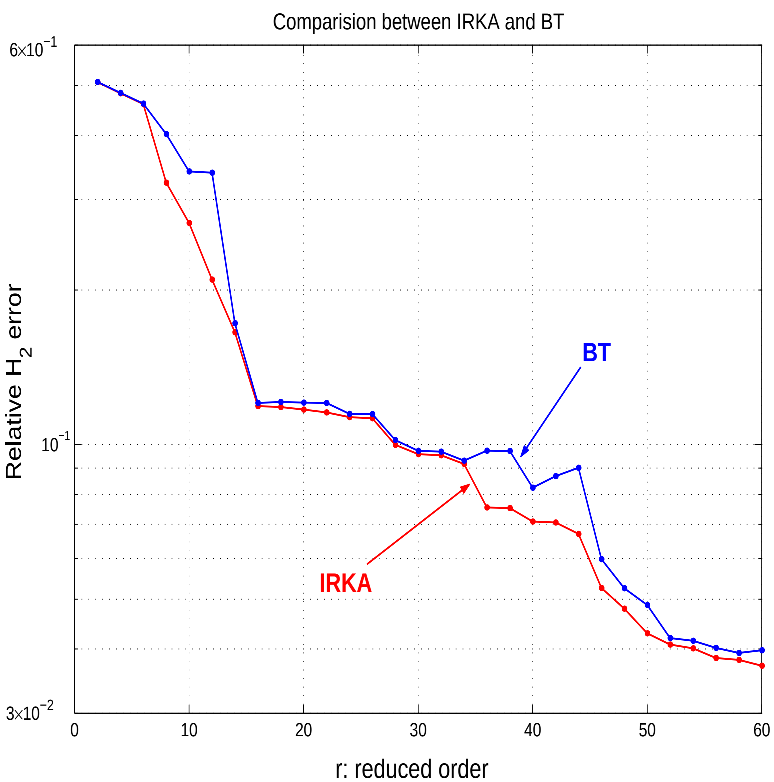
EXTREMELY small order benchmark examples

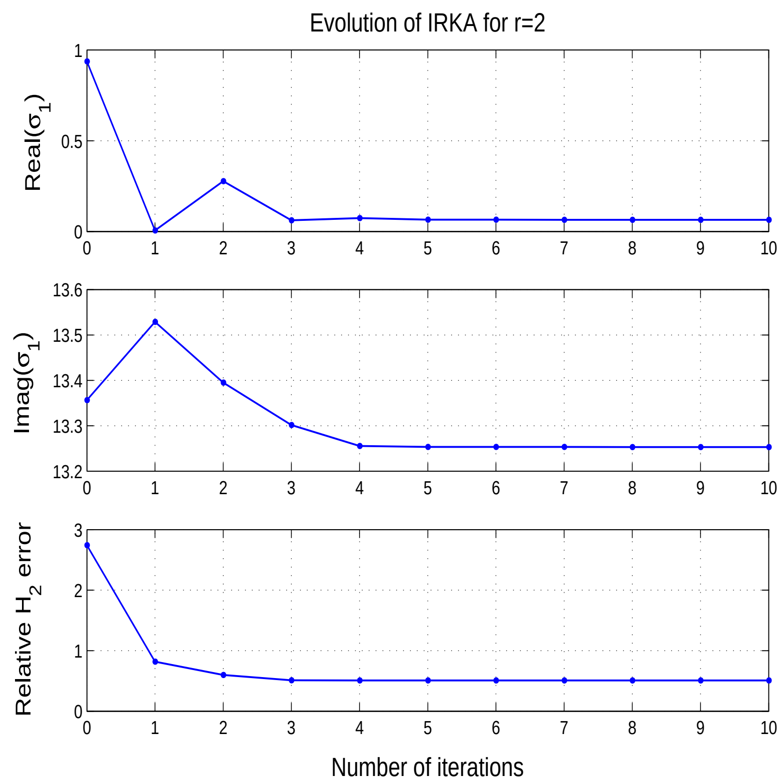
Model	r	**IRKA**	**GFM**	**OPM**	**BTM**
FOM-1	1	4.2683×10^{-1}	4.2709×10^{-1}	4.2683×10^{-1}	4.3212×10^{-1}
FOM-1	2	3.9290×10^{-2}	3.9299×19^{-2}	3.9290×10^{-2}	3.9378×10^{-2}
FOM-1	3	1.3047×10^{-3}	1.3107×19^{-3}	1.3047×10^{-3}	1.3107×10^{-3}
FOM-2	3	1.171×10^{-1}	1.171×10^{-1}	Divergent	2.384×10^{-1}
FOM-2	4	8.199×10^{-3}	8.199×10^{-3}	8.199×10^{-3}	8.226×10^{-3}
FOM-2	5	2.132×10^{-3}	2.132×10^{-3}	Divergent	2.452×10^{-3}
FOM-2	6	5.817×10^{-5}	5.817×10^{-5}	5.817×10^{-5}	5.822×10^{-5}
FOM-3	1	4.818×10^{-1}	4.818×10^{-1}	4.818×10^{-1}	4.848×10^{-1}
FOM-3	2	2.443×10^{-1}	2.443×10^{-1}	Divergent	3.332×10^{-1}
FOM-3	3	5.74×10^{-2}	5.98×10^{-2}	5.74×10^{-2}	5.99×10^{-2}
FOM-4	1	9.85×10^{-2}	9.85×10^{-2}	9.85×10^{-2}	9.949×10^{-1}

- **GFM**: Gradient Flow Method of Yan and Lam [1999]
- **OPM**: Optimal Projection Method of Hyland and Bernstein [1985]
- **BTM**: Balanced Truncation Method of Moore [1981]
- FOM-1: $n = 4$, FOM-2: $n = 7$, FOM-3: $n = 4$, FOM-4: $n = 2$,

ISS 12a Module

- $n = 1412$. Reduce to $r = 2 : 2 : 60$
- Compare with balanced truncation





Part I: Conclusions and Future Work:

- Equivalence of first-order conditions for the $\mathcal{H}_2$ problem
- Iterative Rational Krylov for optimal $\mathcal{H}_2$ reduction
  - First-order conditions while staying in Krylov framework
  - No Lyapunov equations need to be solved
- Good $\mathcal{H}_\infty$ performance as well (Zolatorjov Problem (Beattie [2005])).
- Some open issues remain for convergence and stability.
- Newton's Iteration Formulation
- Application to controller reduction: Gugercin/Antoulas/Beattie [2005]
- Variations that guarantee stability (Gugercin [2005])
- Find another name and acronym better than **IRKA**

Inexact Solves in Krylov-based Model Reduction

- Need for more detail and accuracy in the modeling stage $\Rightarrow$
- System dimension n: $\mathcal{O}(10^6)$ or more $\Rightarrow$
- $(\sigma\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{b}$ cannot be solved directly
- Inexact solves need to be employed in constructing $\mathbf{V}$ and $\mathbf{Z}$
- Questions:
  1. What are the perturbation effects on interpolation?
  2. Robustness with respect to the inexact solves?
  3. What are the effective preconditioning, restarting strategies?
  4. What is the effect on (the optimality of) the reduced model?

- For simplicity, consider the one-sided projection, i.e. $\mathbf{V} = \mathbf{Z}$.
- Let $\hat{\mathbf{v}}_j$ be an inexact solution for $(\sigma_j \mathbf{I} - \mathbf{A})\mathbf{v}_j = \mathbf{b}$

$$(\sigma_j \mathbf{I} - \mathbf{A})\hat{\mathbf{v}}_j - \mathbf{b} = \delta \mathbf{b}_j \quad \text{with} \quad \frac{\|\delta \mathbf{b}_j\|}{\|\mathbf{b}\|} \leq \epsilon$$

- Define $\quad \delta \mathbf{v}_j := \hat{\mathbf{v}}_j - \mathbf{v}_j = (\sigma_j \mathbf{I} - \mathbf{A})^{-1} \delta \mathbf{b}_j$, and

$$\hat{\mathbf{K}} := \left[\, (\sigma_1 \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta \mathbf{v}_1, \quad \cdots \quad (\sigma_r \mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \delta \mathbf{v}_r \,\right].$$

- Inexact Krylov-based reduced model obtained by

$$\mathbf{A}_r = \widehat{\mathbf{V}}^T \mathbf{A} \widehat{\mathbf{V}}, \quad \mathbf{b}_r = \widehat{\mathbf{V}}^T \mathbf{b}, \quad \mathbf{c}_r = \mathbf{c}\widehat{\mathbf{V}}, \quad \text{where} \quad \widehat{\mathbf{V}}^T \widehat{\mathbf{V}} = \mathbf{I}_r.$$

- where $\widehat{\mathbf{V}}$ is an orthogonal basis for Range($\hat{\mathbf{K}}$)

Theorem: Given the above set-up,

$$\begin{aligned}\mathbf{c}_r(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r &= \mathbf{c}(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{b} + \varepsilon_{\text{fwd}} \\ &= \mathbf{c}(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1}\left(\mathbf{b} + \Delta\mathbf{b}_j\right)\end{aligned}$$

where

$$\begin{aligned}\varepsilon_{\text{fwd}} &= \mathbf{c}\left[(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1} - \mathbf{V}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{V}^T\right]\delta\mathbf{b}_j. \\ \Delta\mathbf{b}_j &= [\mathbf{I}_n - (\sigma_j\mathbf{I}_n - \mathbf{A})\mathbf{V}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{V}^T]\delta\mathbf{b}_j.\end{aligned}$$

- ε_{fwd}: Forward error, $\Delta\mathbf{b}_j$: Backward error
- How well $\mathbf{V}(\sigma_j\mathbf{I}_r - \mathbf{A}_r)^{-1}\mathbf{V}^T$ approximates $(\sigma_j\mathbf{I}_n - \mathbf{A})^{-1}$
- Expect optimal model to be robust with respect to inexact solves.
- Same analysis valid for the two-sided projection as well.

- **GMRES**:
  1. The same Krylov subspace for each $(\sigma_j \mathbf{I} - \mathbf{A})\mathbf{v}_j = \mathbf{b}$
     $\mathbf{A}\mathbf{W}_k = \mathbf{W}_{k+1}\widetilde{\mathbf{H}}_k \Rightarrow \min \left\| \sigma_j \tilde{\mathbf{I}} - \widetilde{\mathbf{H}}_k - \|\mathbf{b}\|\mathbf{e}_1 \right\|$
  2. $\mathrm{Span}\{\mathbf{v}_j\}_{j=1}^r$ is important, rather than each $\mathbf{v}_j$
     $\Longrightarrow \min_{\mathbf{x} \perp \hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_\ell} \|(\sigma_{\ell+1}\mathbf{I} - \mathbf{A})\mathbf{x} - \mathbf{b}\|$
  3. Two-sided case: BiCG, ...

- **Preconditioning**:
  1. If σ_j is *close* to σ_{j+1}, can re-use preconditioners for different linear systems
  2. Cost of recomputing vs cost of using a close-by preconditioner

Inexact IRKA (I-IRKA)

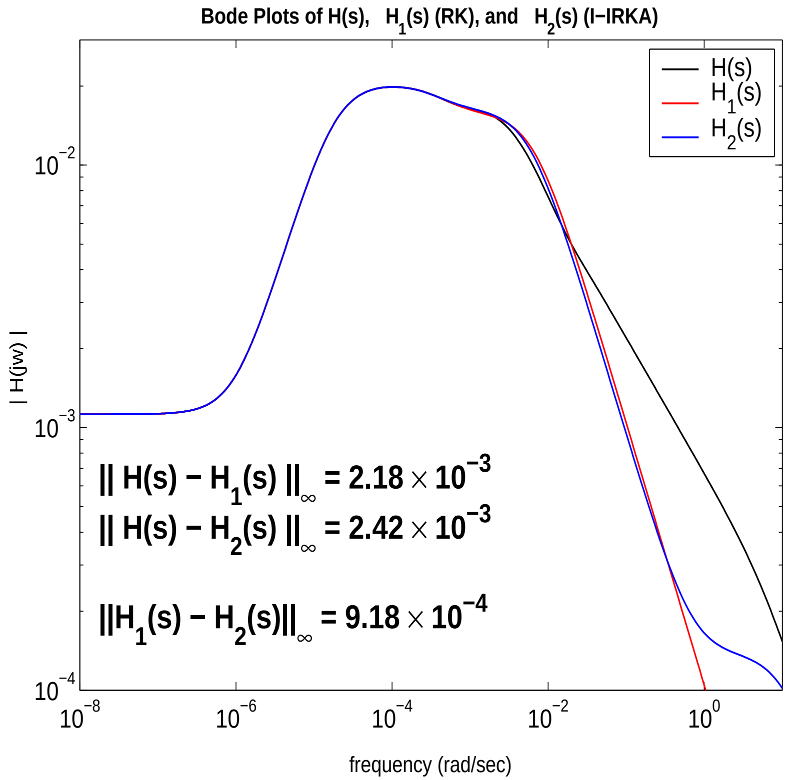
- IRKA requires solving $2r$ linear systems at each step
  $\Rightarrow$ Expensive if $n = \mathcal{O}(10^6)$
- Recall: $\{\sigma_j\}$ converge fast

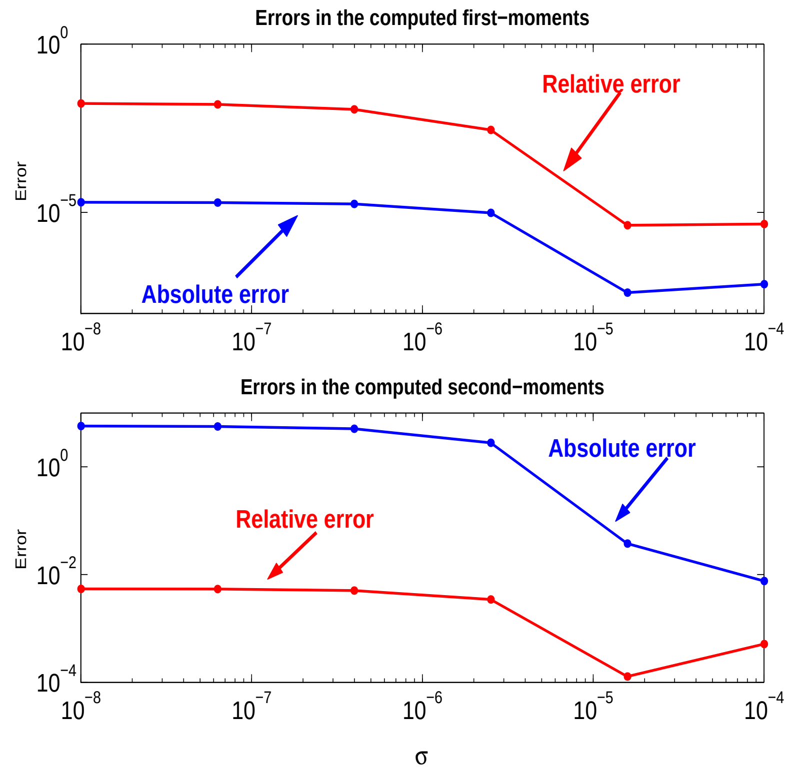
$$\Downarrow$$

- Use the solution from the previous step as an initial guess for the next step
- Expect faster convergence for a fixed tolerance value
- Optimal reduced model: Expect robustness

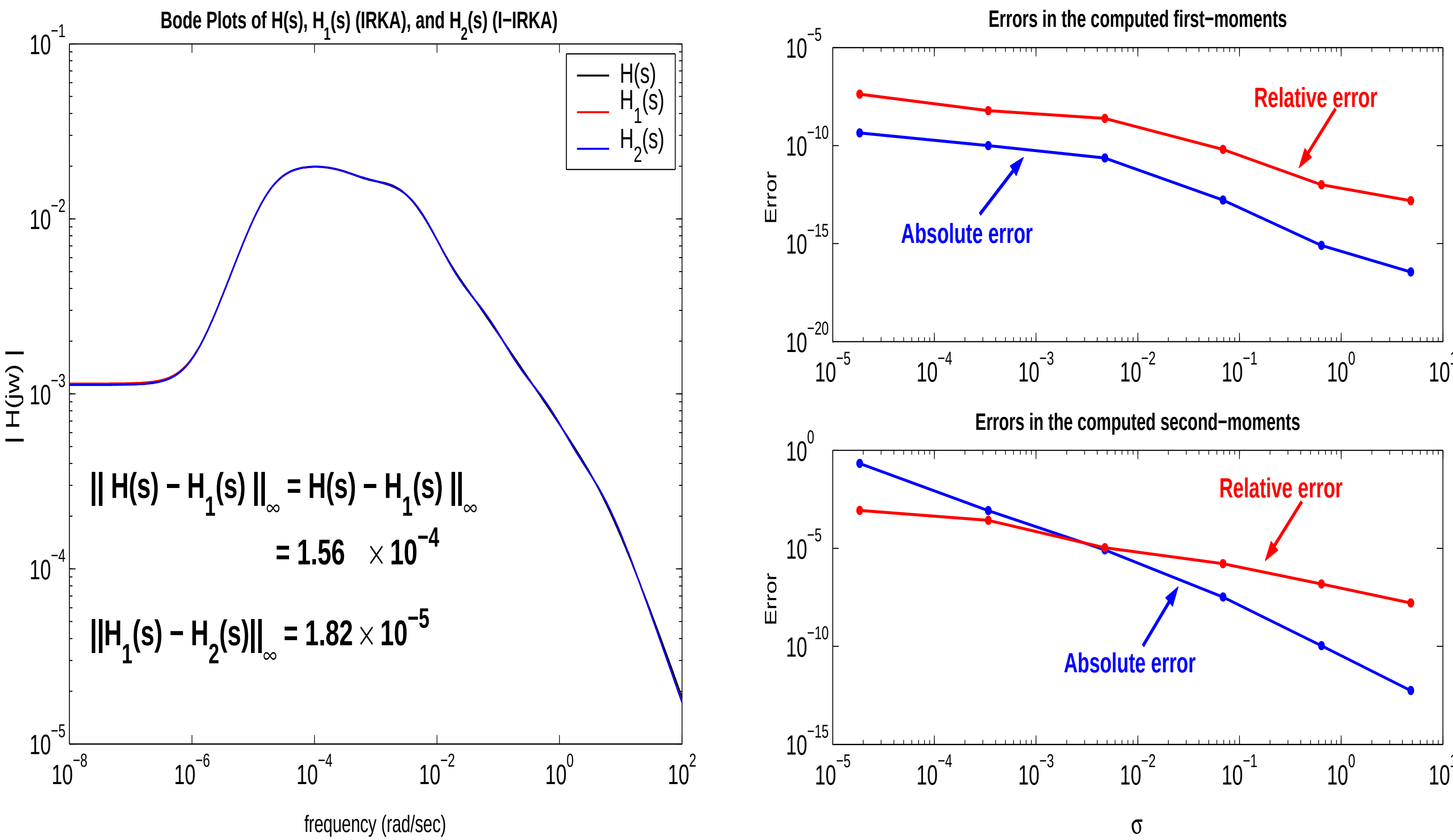
Example: Optimal Cooling of Steel Profiles (P. Benner)

- $\mathbf{G}(s) = \mathbf{c}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$, $n = 20,209$
- Bad shift selection: $\sigma_i = logspace(-8, -4, 6)$
- $r = 6$ via Rational Krylov (**RK**) and Inexact-**RK** (**I-RK**).
- **I-RK** uses GMRES with $tol = 10^{-5}$

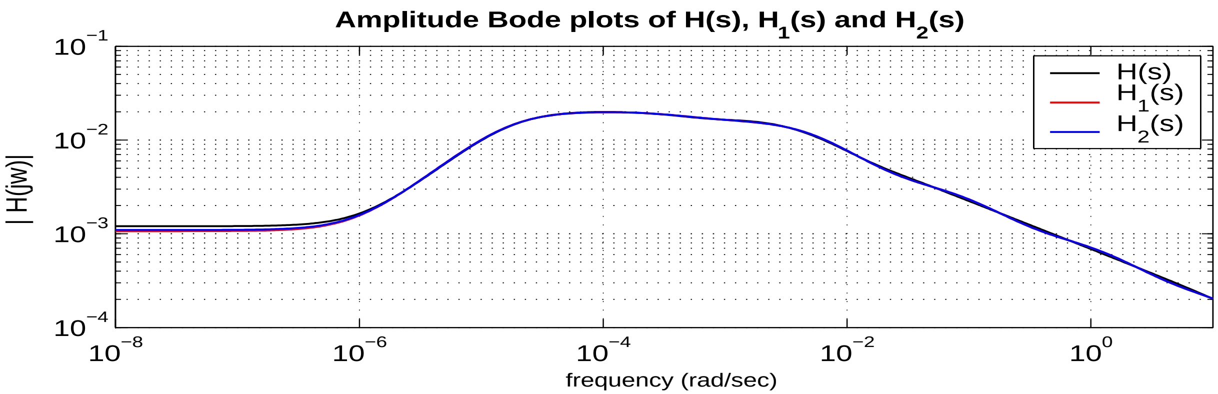


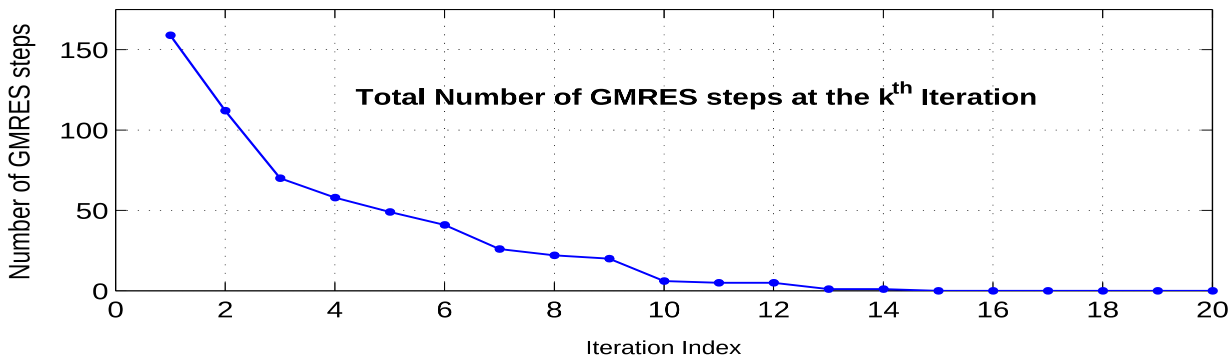


- Optimal $\{\sigma_i\}$ obtained via **IRKA**
- Use these $\{\sigma_i\}$ in **I-RK**.
- **I-RK** uses GMRES with $tol = 10^{-4}$



- Same model with $n = 79,841$ (Finer discretization)
- $r = 6$ via **IRKA** and $\mathbf{I-IRKA}$ ($tol = 5 \times 10^{-5}$)
- **IRKA**: Initial guess from the previous step



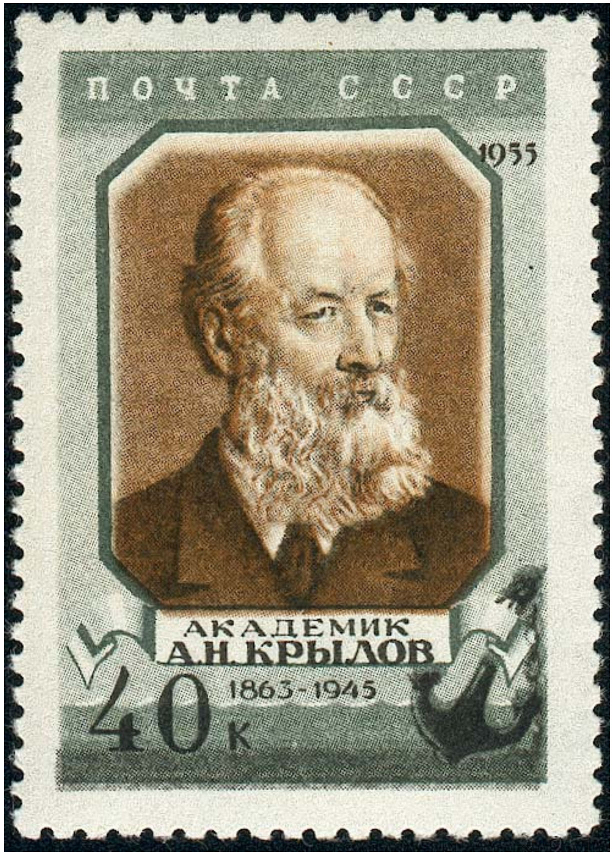


- $\|\mathbf{H}(s) - \mathbf{H}_1(s)\|_\infty = \|\mathbf{H}(s) - \mathbf{H}_2(s)\|_\infty = 6.01 \times 10^{-5}$, $\|\mathbf{H}_1(s) - \mathbf{H}_2(s)\|_\infty = 3.01 \times 10^{-5}$.

Part II: Conclusions and Future Work

- $n >> 10^6$: Forces usage of Inexact Solves in Krylov-based reduction
- Perturbation effects:
  - Backward and forward error analysis framework
  - *Good/Optimal* shift selection robust with respect to inexact solves
  - **I-IRKA**
    - (Locally) optimal reduced models for $n > 10^6$ without user intervention
    - Acceleration strategies
- Open issues:
  - Global $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ perturbation effects
  - Modifications to GMRES, effective preconditioning strategies
  - Scalable parallel versions
    - A large-scale easy-to-use model reduction toolbox
    - Modify the algorithms to fit into the framework of, e.g., Trilinos
    - Implementation on Virginia Tech.-System X

Alexei Nikolaevich Krylov



http://members.tripod.com/jeff560/

[1931] *"On the numerical solution of the equation by which, in technical matters, frequencies of small oscillations of material systems are determined"*
to compute the characteristic polynomial coefficients.

Controller reduction for large-scale systems

- Consider an n^{th} order plant $\mathbf{G}(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$

- n_{κ}^{th} order stabilizing controller: $\mathbf{K}(s) = \mathbf{c}_K(s\mathbf{I} - \mathbf{A}_K)^{-1}\mathbf{b}_K + \mathbf{d}_K$

- LQG, $\mathcal{H}_\infty$ control designs $\quad \Rightarrow \quad n_\kappa = n \quad \Rightarrow$
  (i) *Complex hardware* (ii) *Degraded accuracy*
  (iii) *Degraded computational speed*

- Obtain $\mathbf{K}_r(s)$ of order $r \ll n_\kappa$ to replace $\mathbf{K}(s)$ in the closed loop.

Controller reduction via frequency weighting

- Small *open loop error* $\|K(s) - K_r(s)\|_\infty$ not enough. $\Rightarrow$
- Minimize the weighted error:

$$\boxed{\|W_o(s)(K(s) - K_r(s))W_i(s)\|_\infty .}$$

- How to obtain the weights $W_o(s)$ and $W_i(s)$?
- If $\mathbf{K}(s)$ and $\mathbf{K}_r(s)$ have the same number of unstable poles *and if*

$$\begin{aligned} \left\|[K(s) - K_r(s)]G(s)[I + G(s)K(s)]^{-1}\right\|_\infty &< 1, \text{ \textit{or}} \\ \left\|[I + G(s)K(s)]^{-1}G(s)[K(s) - K_r(s)]\right\|_\infty &< 1, \end{aligned} \Longrightarrow$$

$$\Longrightarrow \quad \mathbf{K}_r(s) \text{ stabilizes } \mathbf{G}(s).$$

- For stability considerations:

$$W_i(s) = I \quad \text{and} \quad W_o(s) = [I + G(s)K(s)]^{-1}G(s) \quad \text{or}$$
$$W_o(s) = I \quad \text{and} \quad W_i(s) = G(s)[I + G(s)K(s)]^{-1}.$$

- To preserve closed-loop performance:

$$W_i(s) = [I + G(s)K(s)]^{-1} \text{ and } W_o(s) = [I + G(s)K(s)]^{-1}G(s).$$

- Solved by frequency-weighted balancing (Anderson and Liu [1989], Schelfhout and De Moor [1996], Varga and Anderson [2002]).

- Requires solving two Lyapunov equations of order $n + n_\kappa$.

$$\mathbf{A}_i\mathcal{P} + \mathcal{P}\mathbf{A}_i^T + \mathbf{b}_i\mathbf{b}_i^T = 0, \quad \mathbf{A}_o^T\mathcal{Q} + \mathcal{Q}\mathbf{A}_o + \mathbf{c}_o^T\mathbf{c}_o = 0,$$

- $\mathbf{A}_i, \mathbf{b}_i$: $\mathbf{K}(s)W_i(s)$, $\quad \mathbf{A}_o, \mathbf{c}_o$: $W_o(s)\mathbf{K}(s)$

- Balance $\mathcal{P}$ and $\mathcal{Q}$.

Controller-reduction via Krylov Projection

- How to modify **IRKA** for the controller reduction problem?

- Let $W_i(s) = I$ and $W_o(s) = [I + G(s)K(s)]^{-1} G(s) \quad \Rightarrow$

- $\underbrace{\mathbf{A}_K \mathcal{P} + \mathcal{P} \mathbf{A}_K^T + \mathbf{b}_K \mathbf{b}_K^T = 0}_{\text{unweighted Lyapunov eq.}} \qquad \underbrace{\mathbf{A}_w^T \mathcal{Q} + \mathcal{Q} \mathbf{A}_w + \mathbf{c}_w^T \mathbf{c}_w = 0.}_{\text{weighted Lyapunov eq.}}$

- $\mathbf{Z} = \mathcal{K}(\mathbf{A}^T, \mathbf{C}^T, \sigma_i)$, and $\mathbf{V} = \mathcal{K}(\mathbf{A}, \mathbf{B}, \mu_j)$

- $\underline{\mathbf{Z} \text{ and } \sigma_i}$: Reflect $W_o(s)$: the closed-loop information.

  $\boxed{\sigma_i = \jmath w_i \text{ over the region where } W_o(\jmath w) \text{ is dominant}}$

- $\underline{\mathbf{V} \text{ and } \mu_j}$: Obtained in an (optimal) open loop sense.

  $\boxed{\mu_j : \text{From an iterative rational Krylov iteration}}$

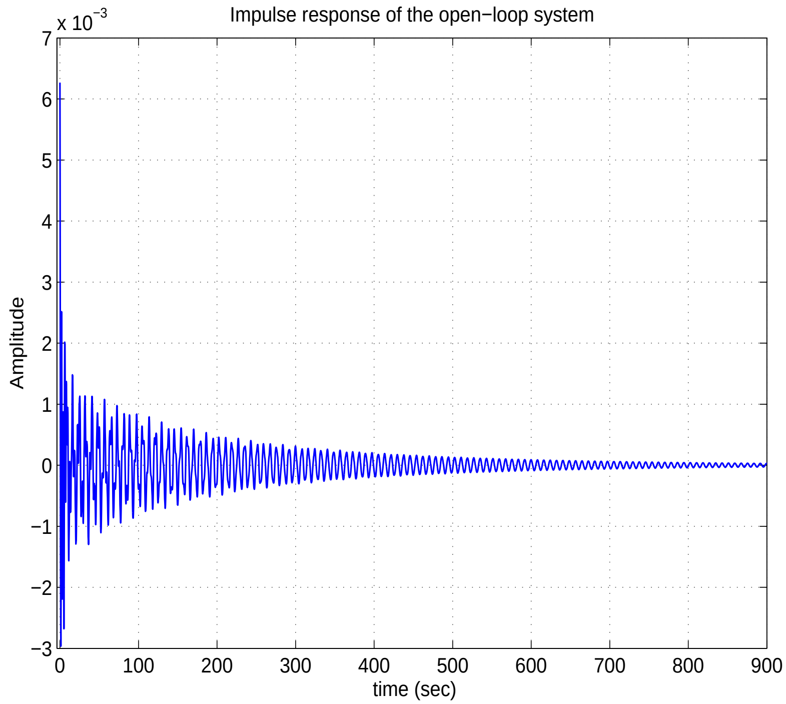
An Iterative Rational Krylov Iteration for Controller Reduction:

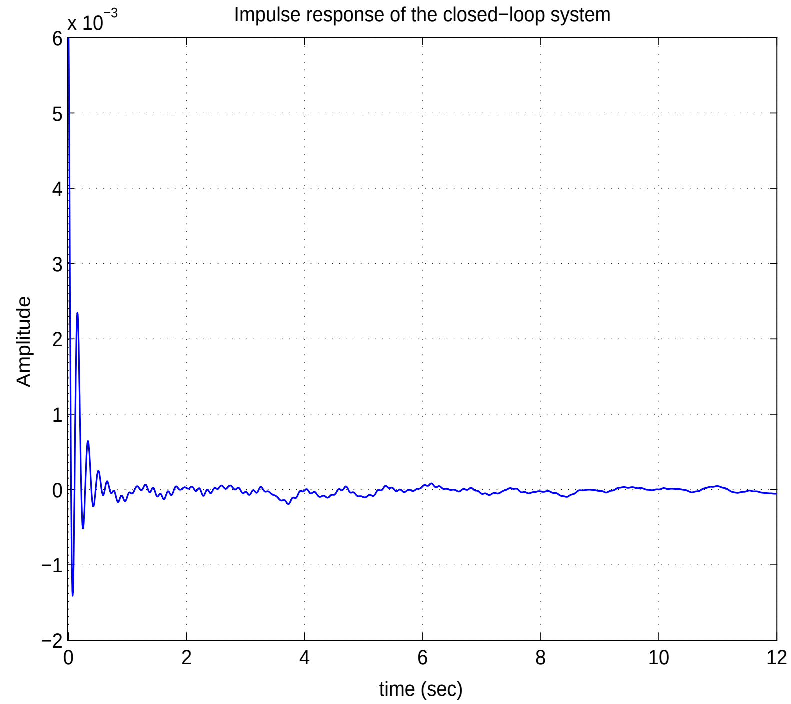
1. Choose $\sigma_i = \jmath w_i$, for $i = 1, \ldots, r$ where w_i is chosen to reflect $W_o(\jmath w)$.
2. $\mathbf{Z} = \text{Span}\left[(\sigma_1 \mathbf{I} - \mathbf{A}_K^T)^{-1}\mathbf{c}_K^T \ \cdots \ (\sigma_r \mathbf{I} - \mathbf{A}_K^T)^{-1}\mathbf{c}_K^T \ \right]$ with $\mathbf{Z}^T\mathbf{Z} = \mathbf{I}_r$.
3. $\mathbf{V} = \mathbf{Z}$
4. while [relative change in μ_j] $> \epsilon$
   (a) $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}_K\mathbf{V}$,
   (b) $\mu_j \longleftarrow -\lambda_i(\mathbf{A}_r)$ for $j = 1, \ldots, r$
   (c) $\mathbf{V} = \text{Span}\left[(\mu_1 \mathbf{I} - \mathbf{A}_K)^{-1}\mathbf{b}_K \ \cdots \ (\mu_r \mathbf{I} - \mathbf{A}_K)^{-1}\mathbf{b}_K \ \right]$ with $\mathbf{Z}^T\mathbf{V} = \mathbf{I}_r$.
5. $\mathbf{A}_r = \mathbf{Z}^T\mathbf{A}_K\mathbf{V}, \qquad \mathbf{b}_r = \mathbf{Z}^T\mathbf{b}_K, \qquad \mathbf{c}_r = \mathbf{c}_K\mathbf{V}$

$$\left.\begin{array}{l}\mathbf{Z} \Rightarrow \mathbf{K}_r(s) \text{ includes the closed} - \text{loop information} \\ \mathbf{V} \Rightarrow \mathbf{K}_r(s) \text{ is optimal in a restricted } \mathcal{H}_2 \text{ sense}\end{array}\right\} \Pi = \mathbf{Z}\mathbf{V}^T$$

International Space Station Module 1R:

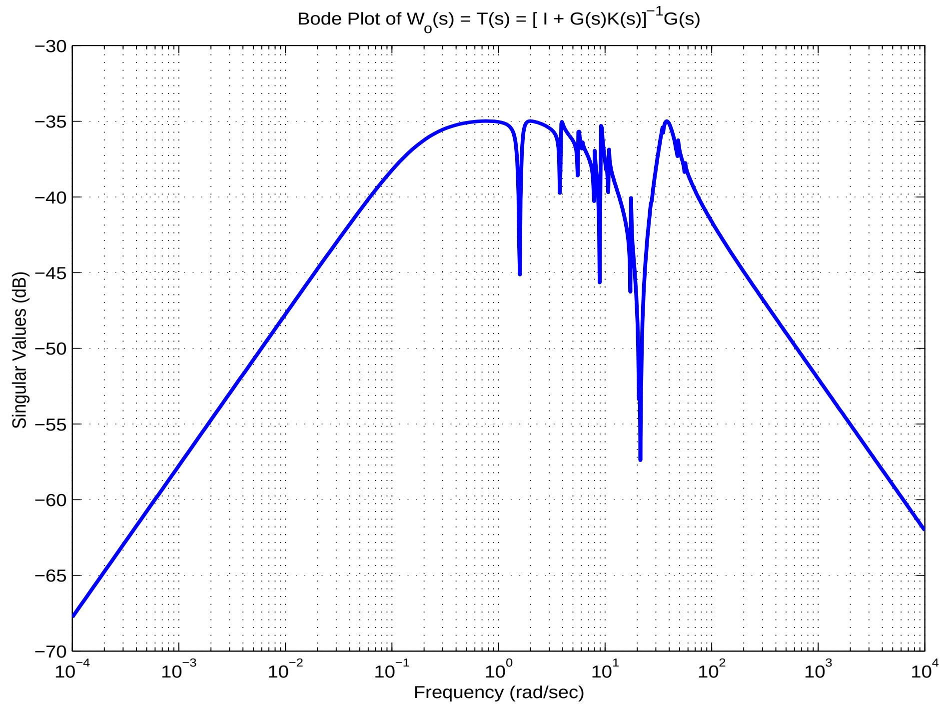
- $n = 270$. $G(s)$ is lightly damped $\Rightarrow$ Long-lasting oscillations.
- $K(s)$ is designed to remove these oscillations. $n_K = 270$.



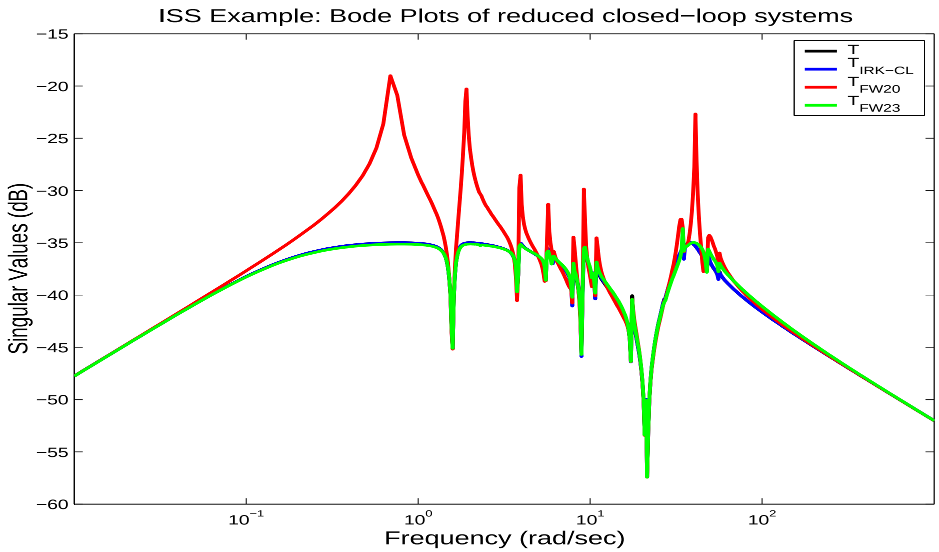


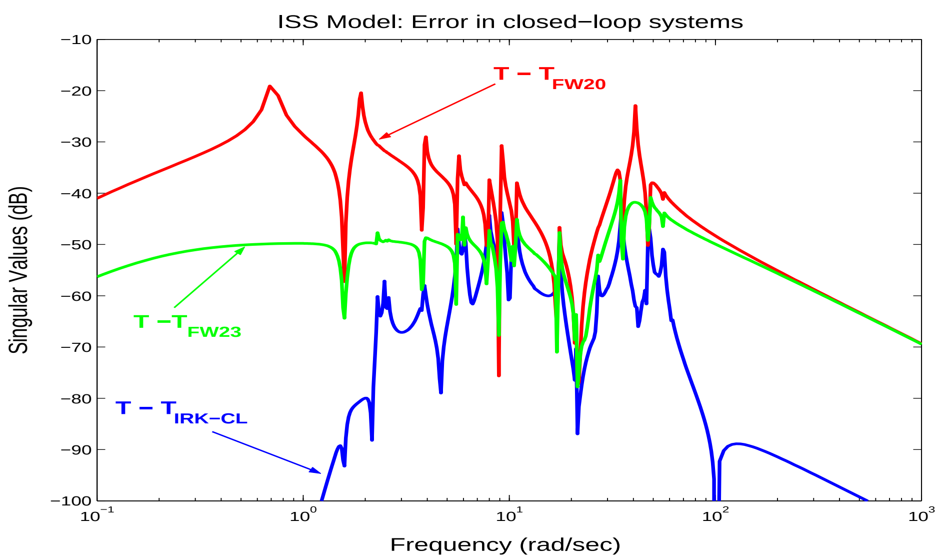
- Reduce the order to $r = 19$ using iterative Rational Krylov and to $r = 23$ using one-sided frequency weighted balancing

- **FWBR**: Frequency-weighted balancing with $W_i(s) = I$ and $W_o(s) = [I + G(s)K(s)]^{-1} G(s)$.
- **IRK-CL**: Iterative Rational Krylov - Closed Loop version: σ_i reflect the weight $W_o(s)$.



- $\sigma_i = \jmath * logspace(-1, 2, 10)$ rad/sec





Relative Errors

	$\mathcal{H}_\infty$ error
$T - T_{\mathrm{FW20}}$	3.88×10^{1}
$T - T_{\mathrm{FW23}}$	5.63×10^{-1}
$T - T_{\mathrm{IRK-CL}}$	1.47×10^{-1}

Relative Errors

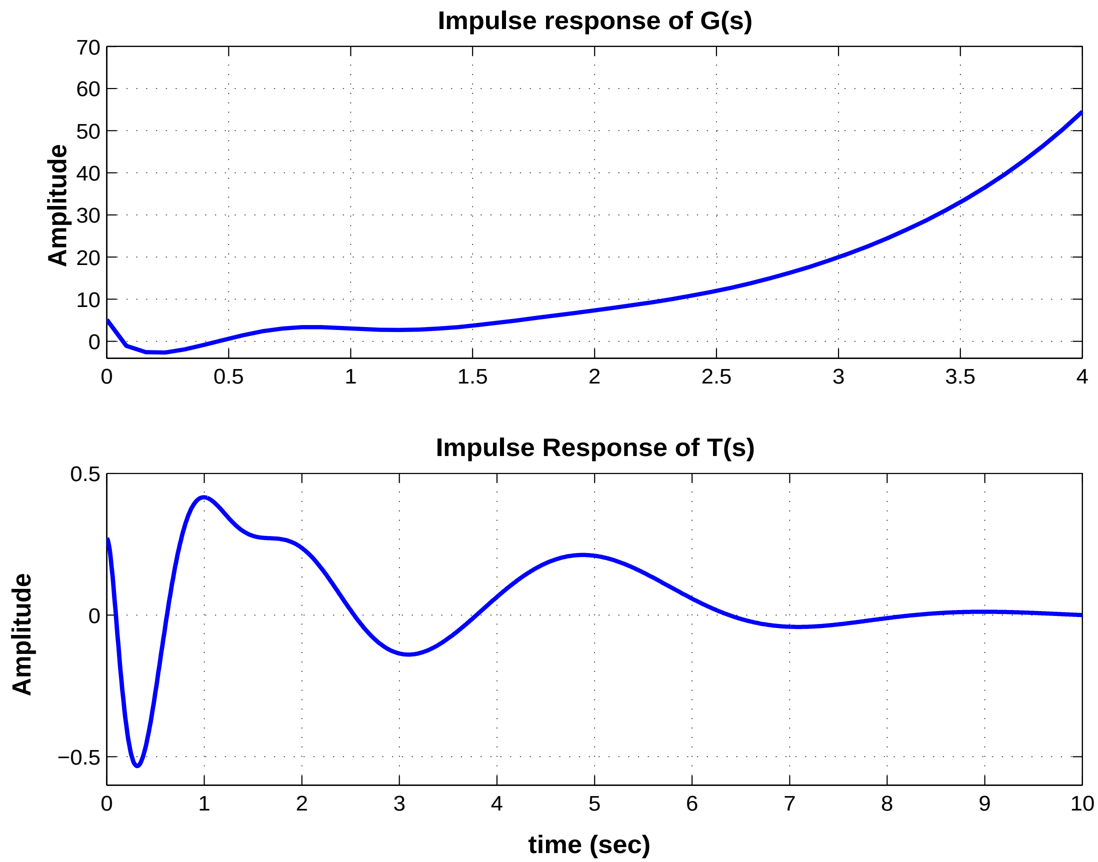
	$\mathcal{H}_2$ error
$T - T_{\mathrm{FW20}}$	3.90×10^{0}
$T - T_{\mathrm{FW23}}$	1.88×10^{-1}
$T - T_{\mathrm{IRK-CL}}$	3.57×10^{-2}

Weighted Errors

	$\mathcal{H}_2$ error
$W_i(K - K_{\mathrm{FW20}})$	$0.984 < 1$
$W_i(K - K_{\mathrm{FW23}})$	$0.416 < 1$
$W_i(K - K_{\mathrm{IRK-CL}})$	$0.365 < 1$

An Unstable Model:

- n=2000. $\mathbf{K}(s)$ of order $n_{\kappa} = 2000$ stabilizes the model.



- $\mathbf{K}(s)$ has four unstable poles.

- Reduce the order to $r = 14$: Stabilizing controller
- $\mathbf{K}_r(s)$ has 4 unstable poles as desired.